

# Exact solution of generalized Dicke models via Susskind-Glogower operators

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## Abstract

We show a right unitary transformation approach based on Susskind-Glogower operators that diagonalizes a generalized Dicke Hamiltonian in the field basis and delivers a tridiagonal Hamiltonian in the Dicke basis that is diagonalized by a set of orthogonal polynomials satisfying a three-term recurrence relation. Our result is used to deliver a closed form analytic time evolution for the case of a Jaynes-Cummings-Kerr model and to study the time evolution of the population inversion, reduced field entropy and Husimi's Q-function of the field for ensembles of interacting two-level systems under a Jaynes-Cummings-Kerr model.

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## I. INTRODUCTION

The Jaynes-Cummings (JC) model is a fundamental building block in quantum optics; it describes the interaction of a qubit with a quantum electromagnetic field under long wave and rotating wave approximations. It is exactly solvable [1] and has proven useful to describe phenomena as Rabi oscillations [2] and collapse and revivals of the atomic inversion [3], just to mention a couple; see [4] for a review on the model. If the number of qubits increases, the model, known as Dicke or Tavis-Cummings model, shows many-body phenomena in the form of a superradiant phase [5]. Dicke model is also exactly solvable [6–8] and is known to show super-fluorescence and amplified spontaneous emission; see [9] for a recent review on the model.

In recent years, a general Dicke Hamiltonian including quadratic self-interactions on both the field and qubit ensemble was introduced to study the effect of the nonlinearities and their relation to the Stark shift, in units of  $\hbar$ ,

$$H = \omega \hat{a}^\dagger \hat{a} + \omega_0 \hat{S}_z + \gamma \left( \hat{a}^{\dagger 2} \hat{a}^2 + \hat{S}_z^2 \right) + g \left( \hat{a} \hat{S}_+ + \hat{a}^\dagger \hat{S}_- \right). \quad (1)$$

An exact solution through quantum inverse methods involving Bethe ansatz was found [10].

The importance of Dicke model and its generalizations lies on its ability to describe more than atoms interacting with the quantized field of a cavity; i.e. lasers. For example, it may describe open dynamical cavity-QED systems [11], ion trap systems [12], circuit-QED systems [13, 14], and Bose-Einstein condensates interacting with classical or quantized electromagnetic fields [15–17].

In this brief report, we present an exact solution, up to the roots of a polynomial, to a more general Dicke Hamiltonian than that in Eq.(1) by including non-identical nonlinear interactions. Our exact solution allows us to recover the well known time evolution of a single two-level system, Jaynes-Cummings dynamics, to verify the validity of our approach. Then, we study the time evolution of different ensemble sizes under identical values of quadratic self-interactions to illustrate the simplicity of our approach and the results it yields; we focus on the dynamics of the population inversion of the two-level system ensemble as well as the dynamics of the entropy and Q-function of the field.

## II. THE MODEL

Let us consider a system composed by an ensemble of  $N$  self-interacting two-level systems, hereby referred to as qubits, in the presence of a quantized field and a Kerr medium described by the Hamiltonian (in units of  $\hbar$ )

$$\begin{aligned} H &= \hat{H}_0 + \hat{H}_I, \\ \hat{H}_0 &= \omega_f \hat{a}^\dagger \hat{a} + \kappa (\hat{a}^\dagger \hat{a})^2 + \chi (\hat{a}^2 + \hat{a}^{\dagger 2}) + \omega_q \hat{J}_z + \frac{\xi}{N} \hat{J}_z^2, \\ \hat{H}_I &= \frac{g}{\sqrt{N}} (\hat{a} + \hat{a}^\dagger) (\hat{J}_+ + \hat{J}_-), \end{aligned} \quad (2)$$

where the  $\mathbf{A}^2 \propto (\hat{a} + \hat{a}^\dagger)^2$  term has not been dropped. The qubits ensemble is described by collective Dicke operators satisfying the  $su(2)$  algebra,  $[\hat{J}_+, \hat{J}_-] = 2\hat{J}_z$ ,  $[\hat{J}_z, \hat{J}_\pm] = \pm \hat{J}_\pm$ , while the annihilation and creation operators for a single mode field satisfy  $[\hat{a}, \hat{a}^\dagger] = 1$ . The rest of parameters are defined as usual, the modified frequency of the field and the two level transition frequency are given by  $\omega_f = \omega_0 - \chi$  and  $\omega_q$ , where  $\omega_0$  is the field frequency, the second order nonlinearity that arises from the  $\mathbf{A}^2$  term by  $\chi$ , the Kerr medium by  $\kappa$ , and the dipole-dipole and ensemble-field couplings by  $\xi$  and  $g$ . Classical dynamics for this Hamiltonian and reductions have been considered in the context of two-mode Bose-Einstein condensates coupled to radiation fields [18–20].

Here, we are going to consider a regime of weak nonlinearities in order to use a squeezed states basis for the field, described by the transformation,

$$\hat{T}_1 = e^{\frac{\chi}{2\omega_f}(\hat{a}^2 - \hat{a}^{\dagger 2})}, \quad \frac{\chi}{\omega_f} \ll 1 \quad (3)$$

to help us get rid of the second order nonlinearity,  $\chi$ . Then, we can introduce a *small rotation* [21],

$$\hat{T}_2 = e^{\frac{\tilde{g}}{\tilde{\omega}_f + \omega_q}(\hat{a} - \hat{a}^\dagger)(\hat{J}_+ + \hat{J}_-)}, \quad \frac{\tilde{g}}{\tilde{\omega}_f + \omega_q} \ll 1. \quad (4)$$

We want to emphasize that this *small rotation* is valid even in the regime where phase transitions appear  $g_c = \sqrt{\tilde{\omega}_f(\omega_q - \xi)}$  [18, 22]. The *small rotation* has an effect similar to that of the rotating wave approximation and leads to a Dicke Hamiltonian including a Kerr medium and dipole-dipole interactions between the qubit ensemble components,

$$\hat{H} = \delta \hat{J}_z + \kappa (\hat{a}^\dagger \hat{a})^2 + \gamma \hat{J}_z^2 + \lambda (\hat{a} \hat{J}_+ + \hat{a}^\dagger \hat{J}_-), \quad (5)$$

where we have moved to a frame defined by the total excitation number  $\hat{N} = \hat{a}^\dagger \hat{a} + \hat{J}_z$  rotating at the frequency of the field and defined the parameters  $\delta = \omega_q - \omega_f + 2\chi^2/\omega_f$ ,  $\gamma = \chi/N$  and  $\lambda = 2g(\omega_f - \chi)(\omega_f^2 - 2\chi^2)/\sqrt{N}\omega_f(\omega_f^2 - 2\chi^2 + \omega_a\omega_f)$ . Note that we have taken the self-interaction nonlinearities  $\kappa$  and  $\xi$  a couple orders of magnitude smaller than the transition frequencies in order to neglect products of couplings and nonlinearities. Also, notice that if Hamiltonian (5) were the starting point, it is trivial to include an Stark effect term,  $\mu\hat{n}\hat{J}_z$ . In the special case of equal self-interactions,  $\kappa = \xi$ , without second order nonlinearity,  $\chi = 0$ , the problem can be transformed into the  $N$ -atom maser including Kerr nonlinearity and Stark shift [10].

In a previous work we implemented a right unitary transformation to solve a quantum Landau-Zener problem for a single two-level system [23]. Implementing an extension of such approach for the time-independent model at hand we obtain an evolution operator with the form

$$\hat{U}(t) = \hat{U}_A(t) \hat{U}_B(t), \quad \hat{U}_x = e^{-i\hat{H}_x t}, \quad (6)$$

where the auxiliary Hamiltonians

$$\hat{H}_A = \sum_{j=-N/2}^{N/2} F(j, \hat{n}) |j\rangle\langle j| + \sum_{j=-N/2+1}^{N/2} G(j, \hat{n}) (|j\rangle\langle j-1| + |j-1\rangle\langle j|), \quad (7)$$

$$\hat{H}_B = \sum_{j=-N/2+1}^{N/2} \sum_{k=0}^{N/2-j} F(j, \hat{n}) \rho_k |j\rangle\langle j| + \sum_{j=-N/2+1}^{N/2} \sum_{k=0}^{-N/2+1+j} G(j, \hat{n}) \rho_k (|j-1\rangle\langle j| + |j\rangle\langle j-1|), \quad (8)$$

where the ket  $|j\rangle$  is a Dicke state and  $\rho_j$  is the density matrix for the pure state of the field with  $j$  photons, are diagonalized in the field basis; i.e., they are given in terms of the photon number functions

$$F(j, \hat{n}) = \kappa \left( \hat{n} - \frac{N}{2} + j \right)^2 + j(\delta + \gamma j), \quad (9)$$

$$G(j, \hat{n}) = \lambda \left[ \frac{N}{2} \left( \frac{N}{2} + 1 \right) - j(j-1) \right]^{1/2} \left[ \hat{n} + 1 + \frac{N}{2} - j \right]^{1/2}. \quad (10)$$

### III. EXACT SOLUTION

Here we want to present a different and simpler approach. For this reason, let us define the right unitary transformation

$$\hat{T} = \sum_{j=-\frac{N}{2}}^{\frac{N}{2}} \hat{V}^{\frac{N}{2}+j} |j\rangle\langle j|, \quad (11)$$

$$\hat{T}\hat{T}^\dagger = \mathbb{1}, \quad (12)$$

$$\hat{T}^\dagger\hat{T} = \mathbb{1} - \sum_{j=-\frac{N}{2}+1}^{\frac{N}{2}} \sum_{k=j}^{\frac{N}{2}+k-1} \rho_k |j\rangle\langle j|, \quad \rho_k = |k\rangle_{ff}\langle k|. \quad (13)$$

Again, the ket  $|j\rangle$  is a Dicke state and  $\rho_j$  is the density matrix for the pure state of the field with  $j$  photons. Then, it is possible to write the generalized Hamiltonian (5) as:

$$\hat{H} = \hat{T}\hat{H}_{SC}\hat{T}^\dagger, \quad (14)$$

where the auxiliar Hamiltonian is given by,

$$\begin{aligned} \hat{H}_{SC} = & \sum_{j=-\frac{N}{2}}^{\frac{N}{2}} f(j, \hat{n}) |j\rangle\langle j| + \\ & \sum_{j=-\frac{N}{2}+1}^{\frac{N}{2}} g(j, \hat{n}) (|j\rangle\langle j-1| + |j-1\rangle\langle j|), \end{aligned} \quad (15)$$

where we have used the notation  $\hat{H}_{SC}$  to bring forward that this Hamiltonian is *semi-classical*-like because it is only expressed in terms of the number operator,

$$f(j, \hat{n}) = \kappa \left( \hat{n} - \frac{N}{2} - j \right)^2 + j(\delta + \gamma j), \quad (16)$$

$$g(j, \hat{n}) = \lambda \left[ \frac{N}{2} \left( \frac{N}{2} + 1 \right) - j(j-1) \right]^{1/2} \left[ \hat{n} + 1 - \frac{N}{2} - j \right]^{1/2}. \quad (17)$$

It is possible to express the dynamics of this model as the evolution operator

$$\hat{U}(t) = e^{-it\hat{H}} = \sum_m \frac{1}{m!} \left( -it\hat{H} \right)^m, \quad (18)$$

where powers of the form  $\hat{H}^m$  are needed. These powers can be obtained by realizing that  $\hat{H}_{SC}\hat{T}^\dagger\hat{T}\hat{H}_{SC} = \hat{H}_{SC}^2$  leads to  $\hat{H}^m = \hat{T}\hat{H}_{SC}^m\hat{T}^\dagger$ . This yields the evolution operator in the reduced form

$$\hat{U}(t) = \hat{T}e^{-it\hat{H}_{SC}}\hat{T}^\dagger. \quad (19)$$

The Hamiltonian  $\hat{H}_{SC}$  is diagonal in the field basis and is symmetric tridiagonal in the Dicke basis; thus it is diagonalizable in the angular momentum basis. The eigenvalues of this Hamiltonian can be found by the method of minors and are given by the roots of the characteristic polynomial

$$p_{N+1}(\nu) = \left[ \nu - f\left(-\frac{N}{2}, \hat{n}\right) \right] p_N(\nu) - g^2 \left(-\frac{N}{2} + 1, \hat{n}\right) p_{N-1}(\nu) \quad (20)$$

with

$$p_0(\nu) = 1, \quad (21)$$

$$p_1(\nu) = \nu - f\left(\frac{N}{2}, \hat{n}\right), \quad (22)$$

$$p_j(\nu) = \left[ \nu - f\left(\frac{N}{2} + 1 - j, \hat{n}\right) \right] p_{j-1}(\nu) - g^2 \left(\frac{N}{2} + 2 - j, \hat{n}\right) p_{j-2}(\nu), \quad j \geq 2 \quad (23)$$

The eigenvectors are calculated from the eigenvalue equation for the Hamiltonian and give

$$|v_j\rangle = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} c_k^{(j)} |k\rangle, \quad \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} |c_k^{(j)}|^2 = 1, \quad (24)$$

where the amplitudes answer to the following recurrence relations,

$$\left[ f\left(\frac{N}{2}, \hat{n}\right) - \nu_j \right] c_{\frac{N}{2}}^{(j)} + g\left(\frac{N}{2}, \hat{n}\right) c_{\frac{N}{2}-1}^{(j)} = 0, \quad (25)$$

$$[f(j, \hat{n}) - \nu_j] c_k^{(j)} + g(j, \hat{n}) c_{k-1}^{(j)} + g(j+1, \hat{n}) c_{k+1}^{(j)} = 0, \quad (26)$$

$$\left[ f\left(-\frac{N}{2}, \hat{n}\right) - \nu_j \right] c_{-\frac{N}{2}}^{(j)} + g\left(-\frac{N}{2} + 1, \hat{n}\right) c_{-\frac{N}{2}+1}^{(j)} = 0. \quad (27)$$

#### IV. EXAMPLES

The time evolution given in the previous section accounts for the full dynamics of the system and helps calculating any given quantity of interest. As an example, we will focus on the time evolution of the reduced density matrix for the field where the initial state is given by a pure state  $|\psi(0)\rangle = \sum_{j=0}^{\infty} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} c_{j,k} |j\rangle_f |k\rangle$ ,

$$\rho_f(t) = \sum_{j,k,l=-\frac{N}{2}}^{\frac{N}{2}} \sum_{p,q=0}^{\infty} c_{p+l-j,j} c_{q+l-k,k}^* U_{l,j} \left( p+l+\frac{N}{2}, t \right) U_{l,k}^* \left( q+l+\frac{N}{2}, t \right) |p\rangle_{ff} \langle q|. \quad (28)$$

The notation  $U_{i,j}(\hat{n}, t) = \left( e^{-t\hat{H}_{SC}} \right)_{i,j}$  is used to describe the components of the *semi-classical* time evolution operator. This allows us to calculate the mean photon number

evolution,

$$\langle \hat{n}(t) \rangle = \sum_{j,k,l=-\frac{N}{2}}^{\frac{N}{2}} \sum_{p=0}^{\infty} p c_{p+l-j,j} c_{p+l-k,k}^* U_{l,j} \left( p+l+\frac{N}{2}, t \right) U_{l,k}^* \left( p+l+\frac{N}{2}, t \right), \quad (29)$$

and in consequence the population inversion  $\langle \hat{J}_z(t) \rangle = \langle \hat{N}(t=0) \rangle - \langle \hat{n}(t) \rangle$ . Another interesting quantity is the purity of the field,

$$P(t) = 1 - \text{Tr } \hat{\rho}_f^2, \quad (30)$$

$$\begin{aligned} \text{Tr } \hat{\rho}^2 = & \sum_{j,k,l,m,n,o=-\frac{N}{2}}^{\frac{N}{2}} \sum_{p,q=0}^{\infty} c_{p+l-j,j} c_{q+o-m,m} c_{q+l-k,k}^* c_{p+o-n,n}^* U_{o,j} \left( p+l+\frac{N}{2}, t \right) \times \\ & U_{o,m} \left( q+o+\frac{N}{2}, t \right) U_{l,k}^* \left( q+l+\frac{N}{2}, t \right) U_{o,n}^* \left( p+o+\frac{N}{2}, t \right), \end{aligned} \quad (31)$$

which is a crude indicator of entanglement in a bipartite system [24, 25] and, often, gives equivalent information to that obtained from von Neumann entropy,

$$\langle \hat{S}_f(t) \rangle = -\text{Tr} [\hat{\rho}_f(t) \ln \hat{\rho}_f(t)], \quad (32)$$

another measure related to bipartite entanglement [26].

For a single qubit,

$$\hat{H} = \kappa \hat{n}^2 + \frac{\delta}{2} \hat{\sigma}_z + \lambda (\hat{a} \hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_-), \quad (33)$$

$$(34)$$

the *semi-classical* Hamiltonian is given by

$$\hat{H} = \begin{pmatrix} \kappa(n-1)^2 + \frac{\delta}{2} & \lambda\sqrt{\hat{n}} \\ \lambda\sqrt{\hat{n}} & \kappa n^2 - \frac{\delta}{2} \end{pmatrix} \quad (35)$$

and it is possible to give a closed form time evolution operator as

$$\hat{U}(t) = \hat{T} e^{-it\hat{H}_{SC}} \hat{T}^\dagger, \quad (36)$$

$$e^{-it\hat{H}_{SC}} = e^{-\frac{it}{2}\kappa[1+2\hat{n}(\hat{n}+1)]} \left\{ \cos \frac{\Omega(\hat{n})t}{2} - \frac{i[\beta(\hat{n})\hat{\sigma}_z + 2\lambda\sqrt{\hat{n}}\hat{\sigma}_x]}{\Omega(\hat{n})} \sin \frac{\Omega(\hat{n})t}{2} \right\}, \quad (37)$$

$$\beta(\hat{n}) = \delta + \kappa(1 - 2\hat{n}), \quad (38)$$

$$\Omega(\hat{n}) = \sqrt{[\beta(\hat{n})]^2 + 4\hat{n}\lambda^2} \quad (39)$$

It is trivial to apply the operator  $\hat{T}^\dagger(\hat{T})$  to any given initial state ket (bra) and then apply the *semi-classical* exponential. Figure 1 shows the time evolution of the mean population

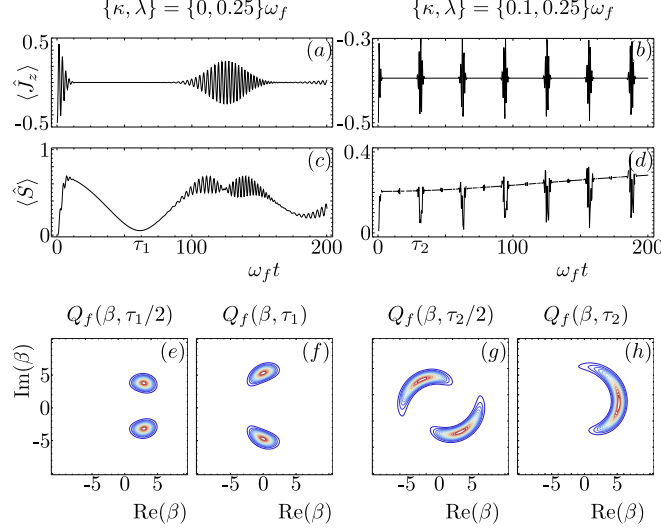


Figure 1. (Color online) Time evolution of the mean population inversion (a,b), reduced field entropy (c,d) and Husimi's Q-function for the field at times equal to half-minimum (e,g) and minimum of entropy (f,h) for a single qubit interacting with a quantized electromagnetic field on resonance,  $\delta = 0$ , under the Jaynes-Cummings model, left column (a,c,e,f), and under a Jaynes-Cummings-Kerr model, right column (b,d,g,h). The initial state for both cases is  $|\psi(0)\rangle = |\alpha\rangle_f |-\frac{3}{2}\rangle$  with  $\alpha = 5$ .

inversion (first row), entropy of the reduced field (second row) and Husimi's Q-function of the field (third row) for a single qubit as given by a Jaynes-Cummings model (left column) and a Jaynes-Cummings-Kerr model (right column). Our results are in accordance with those in the literature [3, 27] and we can proceed to sample the dynamics of ensembles.

For an ensemble of qubits, the task of finding a closed form expression for the time evolution becomes cumbersome but it is possible to numerically diagonalize the *semi-classical* Hamiltonian and implement the time evolution of any given initial state. As an example, we consider the evolution of ensembles of three, Fig. 2, and twenty five, Fig. 3 qubits. The information about the particular initial conditions and parameter values can be found in the figures and their captions. At the time, it is not our goal to report and in-depth analysis of the dynamics of generalized Dicke models but just to present our diagonalization scheme to obtain an exact solution via Susskind-Glogower operators. For this reason we will just briefly comment some basic characteristics of the dynamics. By considering an initial state given by the separable state consisting of a coherent field and the ensemble in its ground state,  $|\psi(0)\rangle = |\alpha\rangle_f | -N/2\rangle$ , it is possible to see that the Dicke model presents strong collapse



and revivals of the population inversion as long as the mean photon number is larger than the number of qubits in the system. A clear collapse of the population inversion is seen in any case studied here, up to  $N \sim \alpha^2$ . The strength of the oscillations in the population inversion diminishes as the number of qubits in the system gets close to the mean photon number of the coherent state but they become ever present at smaller times as we get larger ensemble sizes for a fixed value of the coherent state parameter. Meanwhile, the purity and entropy of such a Dicke model signals an ever-present entangled state between the field and the ensemble as the number of qubits gets close to or equal to the mean number of photons; i.e., the plots change from strong, well defined, unmodulated dips in the functions to a strongly modulated flat-liner close to the value of a mixed reduced density matrix. The Q-function for the reduced field behaves as expected. For  $\alpha \ll N$ ,  $N + 1$  well-defined phase blobs appear and evolve half of them clock-wise and the other half counter-clock-wise as semi-classic time goes by. The revivals in the population inversion are associated to the overlapping of these phase blobs; a stronger revival corresponding to a better overlapping.

On the other hand, when an interacting ensemble of qubits is considered under Dicke-Kerr dynamics, the collapse and revivals of the population inversion are always weak but well defined and periodical. Purity and entropy functions point a return to a quasi-separable state on the first revival for the cases analysed with the number of qubits less or equal to the mean photon number of the field. The mean value of these functions gradually increases with time and some dips appear periodically due to the constructive interference of the wavefunction components leading to revivals in the population inversion. Under Dicke-Kerr dynamics the phase blobs seem heavily defined by the Kerr process and for  $\alpha = 5$  four phase blobs appear and two of them evolve clock-wise while the other two do it counter-clock-wise. This process produces an overlap of two and two of the phase blobs leading to a weak local minima in the purity/entropy but does not register in the population inversion. It is only when the four phase blobs overlap that a pronounced local minima and a revival of the population inversion appears.

## V. CONCLUSION

We have considered a general form of a quantum Rabi Hamiltonian including self-interactions in both the atomic ensemble and the field. We present a transformation that

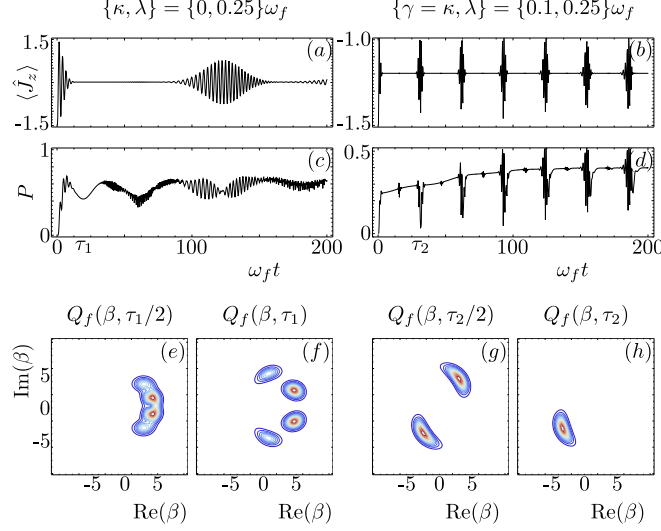


Figure 2. (Color online) Time evolution of the mean population inversion (a,b), reduced field purity (c,d) and Husimi's Q-function for the field at times equal to half-minimum (e,g) and minimum of entropy (f,h) for a quantized electromagnetic field interacting on resonance,  $\delta = 0$ , with three qubits under the Dicke model, left column (a,c,e,f), and with three interacting qubits under a Dicke-Kerr model, right column (b,d,g,h). The initial state for both cases is  $|\psi(0)\rangle = |\alpha\rangle_f |-\frac{3}{2}\rangle$  with  $\alpha = 5$ .

removes the quadratic terms of the annihilation/creation operators and another transformation that acts as a rotating wave approximation in the limit of weak coupling and nonlinearities; these transforms yield a generalized Dicke model. As a side result, we extend a previous result based on Susskind-Glogower operators that gives the exact dynamics of a Jaynes-Cummings model as the product of two evolution operators. Our main result is a different and simpler approach involving Susskind-Glogower operators and right unitary transformations. These allows us to represent our generalized Dicke model as a transformed *semi-classical*-like Hamiltonian which is diagonal in the field basis and tridiagonal in the Dicke basis; thus the diagonalization of this *semi-classical* Hamiltonian is known up to the roots of its characteristic polynomial. The transformed Hamiltonian gives the time evolution of the system and provides access to the dynamics of any quantity of interest.

We use our result to derive a closed analytical form for the time evolution operator of a single qubit interacting with a quantized field in the presence of a Kerr medium, a Jaynes-Cummings-Kerr model. Also, we present the time evolution of the population inversion, reduced field entropy and Husimi's Q-function of the field for ensembles consisting of three

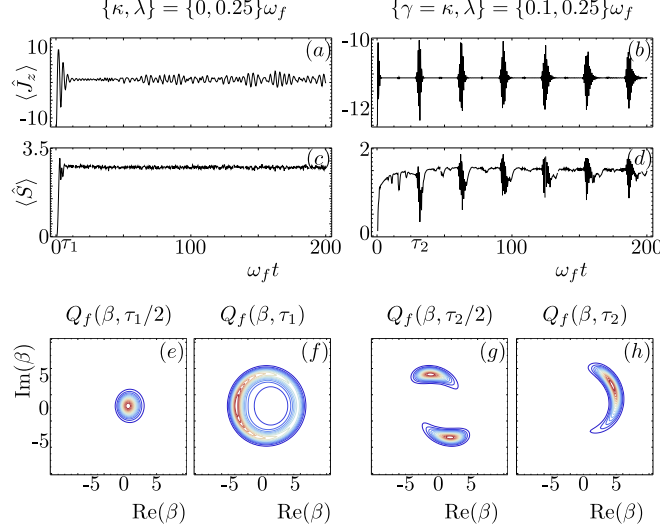


Figure 3. (Color online) Time evolution of the mean population inversion (a,b), reduced field entropy (c,d) and Husimi's Q-function for the field at times equal to half-minimum (e,g) and minimum of entropy (f,h) for a quantized electromagnetic field interacting on resonance,  $\delta = 0$ , with twenty five qubits under the Dicke model, left column (a,c,e,f), and with twenty five interacting qubits under a Dicke-Kerr model, right column (b,d,g,h). The initial state for both cases is  $|\psi(0)\rangle = |\alpha\rangle_f |-\frac{25}{2}\rangle$  with  $\alpha = 5$ .

and twenty-five interacting two-level systems under a Dicke-Kerr model where the interaction and Kerr nonlinearity are equal.

Our approach can follow the dynamics of hundreds and maybe a few thousands of qubits in a simple workstation with efficient programming; this is relevant to the description of micromasers and the study of fields interacting with Bose-Einstein condensates to give just a couple of examples.

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